1 How many standard-legal 60-decks are there?

First consider the following problem: I have two nickles, two dimes, and a quarter. What are the various amounts we can obtain using these coins, and how many ways are there to reach these amounts?

$$\begin{aligned}&(1+x^5+x^{10})+(1+x^{10}+x^{20})(1+x^{25})\\=&1+x^5+2x^{10}+x^{15}+2x^{20}+2x^{25}\\&+2x^{30}+2x^{35}+x^{40}+2x^{45}+x^{50}+x^{55}\end{aligned}$$

The amounts of coins are encoded in the exponent - the number of combinations giving that amount is incoded in the coefficient!

So how many standard decks are there?

Expansion	No. of non-basic cards
Theros	229
Born of the Gods	165
Journey into Nyx	165
Magic 2015	249
Khans of Tarkir	249
Fate Reforged	175
Dragons of Tarkir	249
Magic Origins	252

$$229+2\cdot 165+3\cdot 249+175+252=1733$$

$$[x^{60}](1+x+\cdots+x^{60})^5\cdot(1+x+x^2+x^3+x^4)^{1733}=$$

$$\begin{aligned}&831980451179295583982646415893388048968989997244406350968\\&2716229731542521902944716258103469334310493565022129610000\end{aligned}$$

Problem: Reprints! Minus reprints, we have 1702 cards. Giving

$$\begin{aligned}&287731940719452849449822507195434370455636650082651212 02\\&874694699449036572979127862001991495631917810444646146800\end{aligned}$$

These are example of finite generating functions! Generating functions are ways of expressing sequences $a_0, a_1, a_2, \ldots$ in the way of

$$a_0x^0+a_1x^1+a_2x^2+\ldots$$

2 The Fibonacci Sequence

So moving onto infinite generating functions.

Let $a_0, a_1, a_2, a_3, \ldots$ be the sequence defined by $a_0 = 0$, $a_1 = 1$, $a_{n+2} = a_{n+1} + a_n$. Let $A(x) = \sum a_n x^n$

Then

$$
\begin{aligned}
a_{n+2}x^n =& a_{n+1}x^n + a_n x^n \\
\sum_{n=0}^{\infty} a_{n+2}x^n =& \sum_{n=0}^{\infty} a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n \\
\frac{A(x) - a_0 x^0 - a_1 x^1}{x^2} =& \frac{A(x) - a_0 x^0}{x} + A(x) \\
\frac{A(x) - x}{x^2} =& \frac{A(x)}{x} + A(x) \\
A(x) - x =& xA(x) + x^2 A(x) \\
A(x) - xA(x) - x^2 A(x) =& x \\
A(x)(1 - x - x^2) =& x \\
A(x) =& \frac{x}{1 - x - x^2}
\end{aligned}
$$

Let's do McLaurin series expansion!

$$
\begin{aligned}
A(x) =& \frac{x}{1 - x - x^2} \\
& \implies A(0) = 0 \\
A'(x) =& \frac{(1)(1 - x - x^2) - (x)(-1 - 2x)}{(1 - x - x^2)^2} = \frac{1 + x^2}{(1 - x - x^2)^2} \\
& \implies A'(0) = 1 \\
A''(x) =& \frac{(2x)(1 - x - x^2)^2 - (1 + x^2) \cdot 2(1 - x - x^2)(-1 - 2x)}{(1 - x - x^2)^4} \\
=& \frac{-2x^5 - 2x^4 - 4x^3 - 8x^2 + 4x + 2}{(1 - x - x^2)^4} \\
& \implies A''(0)/2! = 1 \\
A^{(3)}(x) =& \frac{6(x^4 + 6x^2 + 4x + 2)}{(1 - x - x^2)^4} \\
& \implies A^{(3)}(0)/3! = 2
\end{aligned}
$$

So

$$
\frac{x}{1 - x - x^2} = x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots = \sum_{n=0}^{\infty} a_n x^n
$$

where a_n is the nth Fibbanocci number. And that's super duper neato, but is this actually useful?

Well sure! Partial Fraction Decomposition! Let

$$r_+ = \frac{1+\sqrt{5}}{2}, r_- = \frac{1-\sqrt{5}}{2}$$

Then $1 - x - x^2 = (1 - r_+x)(1 - r_-x)$

Then

$$\begin{aligned} \frac{x}{(1-r_+x)(1-r_-x)} &= \frac{A}{1-r_+x} + \frac{B}{1-r_-x} \\ \iff x &= A(1-r_-x) + B(1-r_+x) \\ \iff x &= (A+B) - x(Ar_- + Br_+) \end{aligned}$$

Equating degrees, $B = -A$. Giving

$$\begin{aligned} x =& -xA(r_- - r_+) \\ x =& A(r_+ - r_-)x \\ A =& \frac{1}{r_+ - r_-} \end{aligned}$$

So

$$\frac{x}{1-x-x^2} = \frac{1}{r_+ - r_-}\left(\frac{1}{1-r_+x} - \frac{1}{1-r_-x}\right)$$

dot dot dot... so?

Well Consider this! For a function $\frac{1}{1-ax}$ its MacLaurin series (or generating function) expansion is

$$\sum_{n=0}^{\infty} a^n x^n$$

So then....

$$\begin{aligned} &[x^n]a_0x^0 + a_1x^1 + a_2x^2 + \dots \\ =&[x^n]\frac{x}{1-x-x^2} \\ =&[x^n]\frac{1}{r_+ - r_-}\left(\frac{1}{1-r_+x} - \frac{1}{1-r_-x}\right) \\ =&\frac{1}{r_+ - r_-}\left([x^n]\frac{1}{1-r_+x} - [x^n]\frac{1}{1-r_-x}\right) \\ =&\frac{1}{r_+ - r_-}\left(r_+^n - r_-^n\right) \\ =&\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} \end{aligned}$$

Well then!

3 Binary Strings

For one more example of the power of generating functions, we'll consider binary strings. A binary string is an arbitrary-lengthed string of ones and zeroes. For some set of binary strings S we denote S^* to be the set of binary strings generated by appending elements of S an arbitrary number of time.

$$\begin{aligned}\{0\}^* =& \{\emptyset, 0, 00, 000, 0000, 00000, \dots\} \\ \{000, 101\}^* =& \{\emptyset, 000, 101, 000000, 000101, 101000, 101101, 000000000, \dots\} \\ \{0,1\}^* =& \{\emptyset, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, \dots\}\end{aligned}$$

(So clearly $\{0,1\}^*$ is the set of all binary strings - this makes sense).

We can now work on writing down the generating function for binary strings!

Define our sequence $a_0, a_1, a_2, \dots$ so that a_n is the number of binary strings with whatever constraints there are.

Example: The generating function for binary strings only using zeroes is

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

The generating function for $\{0,1\}^*$ is $1 + (x+x) + (x+x)^2 + (x+x)^3 + \dots = \sum_{n \geq 0} 2^n x^n = \frac{1}{1-2x}$ (which means there are 2^n binary strings of length n - makes sense!

How many binary strings of length n end in a zero? our decomposition of this binary string structure is $\{0,1\}^*0$ The generating function is

$$\left(\frac{1}{1-2x}\right)(x)$$

Then

$$\begin{aligned} & [x^n]\frac{x}{1-2x} \\ =& [x^{n-1}]\frac{1}{1-2x} \\ =& 2^{n-1}\end{aligned}$$

Finally, how many binary strings begin with 1, and are such that any run of zeroes has even length? The decomposition is $1\{1,00\}^*$ which ensures that any block of zeroes has even length. Then our generating function should look something like

$$x\left(1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \dots\right)$$

Which we can write as $x\left(\frac{1}{1-(x+x^2)}\right)$ or

$$\frac{x}{1-x-x^2}$$

which we already found to be the Fibonacci numbers.